

Matrix Model and Elliptic Curve

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Abstract

Solution to the reduced matrix model of IKKT type is studied with non-zero fermion fields. A suggestion is made that our universe is made of rational numbers rather than being a continuum. To substantiate this proposal, the reduced Yang-Mills equation is written in the form of an elliptic curve. The normalization of the solution can be expressed in terms of the Weierstrass function generically or in terms of the Dedekind function in the case of 3-brane. A way to define the gravitational field in the matrix model is proposed with some new interpretation of the cosmological constant. The (first) quantization of the system is done within the framework of non-commutative geometry.

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1 Introduction

Some time ago, two groups, independently of each other, (ref. [1, 2]) made a proposal to formulate the string theory in terms of matrix models. One [1] deals with the M-theory in the light cone gauge and the other [2] defines the IIB string theory in terms of the reduced matrix model [3]. Here, I follow the latter formulation but without being restricted to the IIB model.

The purpose of this paper is to point out a simple fact that the matrix model can be considered within the framework of number theory in a rather elementary way. The motivation of the study is the following.

First, the number of the matrix elements of the matrix models will be infinite but obviously it is not continuous. A conventional wisdom is to take some continuous limit of the matrix model such as to take $N \rightarrow \infty$ and $R \rightarrow \infty$ with N/R fixed to obtain the desired space time continuum. If we take the matrix model as it is as the fundamental theory of space-time without taking the continuum limit, a new insight into the problem emerges. Space-time is composed of integers or at most rational numbers if it is a dense set, as is usually conceived. To give a substance to this view, we give a new meaning to the reduced Yang-Mills equation. Crudely speaking, the equation reads,

$$A \otimes A \otimes A = \psi \times \psi. \quad (1)$$

This is reminiscent of the so-called elliptic curve [4] in the number theory which can be written in general as

$$ax^3 + bx^2 + cx + d = y^2. \quad (2)$$

It is important that we retain the fermion term (classical De Broglie field) in (1) to make a comparison. Since A and ψ in (1) are infinite matrices with the product symbols having a specific meaning, we are treating here an infinite set of coupled elliptic curves as will be shown later in this paper. We assert that the coefficients of these equations are integers or rational numbers and the solutions are also rational numbers. This view enables us to utilize the whole machineries of number theory developed over the last centuries.

For example, we are interested in the number of solutions to equation (2). A zeta function corresponding to equation (2) can be defined to study this problem. In the simple case of integer coefficients, this can be defined as;

$$\zeta(z) = \prod_p (1 - a_p p^{-z} + p^{1-2z})^{-1}, \quad (3)$$

where $a_p = p - N_p$ with N_p the number of solutions of (2) between 0 and p . The product is over all the prime numbers p except for those p which give rise to a multiple root when (2) is considered modulo p .

In the case of equation (1), we assume that the infinite matrix A is composed of clusters of finite matrices and we define zeta function for each of them. If the zeta function for a certain cluster satisfies $\zeta(1) = 0$, we see that we have infinite number of rational number solutions to this cluster due to Birch and Swinnerton-Dyer conjecture [4]. This is almost needed since we have to solve equation (1) together with the other equation of motion which looks symbolically as $A \otimes \psi = 0$. As is well known the elliptic

curve (2) is nothing but a two torus and the relation of (x, y) to the complex coordinate on the upper half plane is given by;

$$x = \wp(z), \quad y = \frac{1}{2}\wp'(z), \quad \text{with } \wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3, \quad (4)$$

where $\wp(z)$ stands for the Weierstrass function. The elliptic curve appeared in physics in different context. Notably, in the analysis of modular space of $N = 2$ supersymmetric gauge theory (Seiberg and Witten [5]), the vacuum values of the fields are defined on the elliptic curve with coordinate of the moduli space u being the zero of $\wp'(z)$. There exist many papers which deal with elliptic curves in various contexts of physics but these are not referred to here since they are not relevant to the present work [6]. In obtaining the zeta function for the equation (1) the Shimura conjecture (or Taniyama-Shimura conjecture) [7] might be helpful. This famous conjecture was proven by A. Wiles [8] in a special case which is needed for the proof of Fermat's theorem and in its full generality by Ch. Breuil, B. Conrad, F. Diamond and R. Taylor in 1999 [9]. The conjecture says that the zeta function (3) is modular.

In this context, what I am trying to propose in this paper is to assert that equation (1) should be considered as modular curve. This naturally leads to the space-time which is composed of field of rational numbers or its algebraic extension.

One of the main research goals of number theory is to generalize the Shimura conjecture to the zeta functions of higher degrees and with an extended automorphism. This so-called Langlands program [9] has a geometric counter part and is being extensively studied in connection with string theory or chiral theory [10, 11].

The organization of the paper is the following: Section 2 prepares some necessary machinery such as the description of the fermion de Broglie fields. The cluster of the matrix will also be defined. Section 3 is devoted to the analysis of equation (1) with the proof that the scale of the classical fields is given by the Weierstrass function or by the Dedekind function. In section 4, a method is proposed to introduce the gravitation into the matrix models. Section 5 is devoted to the problem of (first) quantization. The conjugate variables are introduced as the Dirac operators of the non-commutative geometry. Section 6 is for the brief concluding remarks.

2 Some Preparations

The action of the matrix model is written in such a way to accommodate the non-associative case:

$$L = -\frac{1}{4}\text{Tr} \{[A_\mu, A^\nu][A^\mu, A_\nu]\} - \frac{g}{2}\text{Tr} \{ \bar{\Psi}(A_\mu \Gamma^\mu \Psi) - (A_\mu \bar{\Psi} \Gamma^\mu) \Psi \}, \quad (5)$$

where A_μ is a 10-dimensional Minkowski vector and Ψ is a 10-dimensional 32-component spinor. Depending on whether we take the IIB case or the IIA case, the expression for the Ψ is different.

For the IIB case, we have,

$$\Psi = \left\{ (x + x_{a,b} \Gamma^{a+} \Gamma^{b+} + y B_1) + \Gamma^{0+} (x_a \Gamma^{a+} + y_a \Gamma^{a+} B_1) \right\} \zeta. \quad (6)$$

Here the notation is that of Polchinski [12]:

$$\Gamma^{0\pm} = \frac{1}{2}(\pm\Gamma^0 + \Gamma^1), \quad \Gamma^{a\pm} = \frac{1}{2}(\Gamma^{2a} \pm \Gamma^{2a+1}), \quad (a = 1, 2, 3, 4)$$

$$B_1 = \Gamma^3\Gamma^5\Gamma^7\Gamma^9,$$

Γ^{a+} are creation operators and Γ^{a-} are annihilation operators including $a = 0$,

ζ is a ground state with $\Gamma^{a+}\zeta = 0$ for $a = 0, 1, 2, 3, 4$,

$x, x_{a,b}, y, x_a$ and y_a are all complex numbers.

This case corresponds to two 16-component Majorana-Weyl spinors of same chirality.

For the IIA case we have similarity:

$$\begin{aligned} \Psi = & \left\{ (x + x_{a,b}\Gamma^{a+}\Gamma^{b+} + yB_1) + \Gamma^{0+}(x_a\Gamma^{a+} + y_a\Gamma^{a+}B_1) \right. \\ & \left. + \Gamma^{0+}(x' + x'_{a,b}\Gamma^{a+}\Gamma^{b+} + y'B_1) + (x'_a\Gamma^{a+} + y'_a\Gamma^{a+}B) \right\} \zeta. \end{aligned} \quad (7)$$

The Majorana condition reads,

$$\begin{aligned} x^* &= y, \quad x_{a,b}^* = -\varepsilon_{abcd}y_{c,d}, \quad x_a^* = y_a, \\ x'^* &= y', \quad x'_{a,b} = -\varepsilon_{abcd}y'_{c,d}, \quad x'_a = y'_a \quad \text{and} \quad \zeta^* = \zeta. \end{aligned}$$

We have two Majorana-Weyl spinors of opposite chirality in this case. The variables x 's and y 's are all infinite component matrices as is A_μ .

The next task is to define the cluster and to express the infinite matrix as:

$$A_\mu = \begin{pmatrix} A_\mu^{(1)} & & & & \\ & A_\mu^{(2)} & & & \\ & & A_\mu^{(3)} & & \\ & & & \dots & \\ & & & & \dots \end{pmatrix} \quad (8)$$

We assume that each finite dimensional cluster matrix (A_μ, x, y, \dots) to be expanded as (suppressing the cluster index):

$$A_\mu = A_{i,\mu} \lambda_i,$$

where λ_i satisfies the following multiplication rules:

$$\lambda_i \lambda_j = (if_{ijk} + d_{ijk})\lambda_k + (g_{ij} + ie_{ij})I. \quad (9)$$

The only conditions for the real parameters f, d, g and e are that f and e are antisymmetric with respect to the interchange of the first two suffices and d and g are symmetric. In fact, the reality of $\bar{\Psi}\Gamma^\mu\Psi$ imposes further condition that either of the following two cases be satisfied,

- (1) $e_{ij} = 0$, d_{ijk} is cyclic symmetric in i, j, k ,
- (2) $e \neq 0$, $d = 0$.

Coexistence of f and e (case (2)) corresponds to the non-associative case. In this case, the matrix representation exists using the non-associative algebra itself as the representation space but the multiplication rule (9) should be applied rather than the regular matrix rule. We also assume, $T_r(\lambda_i) = 0$, $T_r(I) = a$ (some finite number).

The motivation to generalize the matrix model to the non-associative case stems from the recent interest in the non-associative algebra which arose in connection with the M2-brane analysis initiated by J. Bagger and N. Lambert [13].

The simplest non-associative example of equation (9) is given by the following “generalized Pauli spin”:

$$\sigma_i \sigma_j = i \varepsilon_{ijk} \sigma_k + (\delta_{ij} + i \kappa \varepsilon_{ij}) I. \quad (10)$$

For $x = a_i \sigma_i + a_0 I$, $y = b_i \sigma_i + b_0 I$, $z = c_i \sigma_i + c_0 I$, we have,

- (1) Jordan condition: $(xy)z - x(yz) = 0$,
- (2) symmetrized associator: $[x, y, z] = 4i\kappa \left\{ (\vec{a} \times \vec{b} \cdot \vec{I}) \vec{c} + (\vec{b} \times \vec{c} \cdot \vec{I}) \vec{a} + (\vec{c} \times \vec{a} \cdot \vec{I}) \vec{b} \right\} \vec{\sigma}$,

where \vec{I} is a vector with a unit length of all three components and $\vec{a} = (a_1, a_2, a_3)$ etc.

Since we regard the each cluster to correspond to a small portion of space-time coordinate, the order of assigning each solution to a cluster should not matter. This will be regarded as the transformation which replaces the general coordinate transformation as will be discussed in section 4.

3 Equations of Motion

From the action (5) we get the following equations of motion for each cluster:

$$4G_{ij;i'j'} A_j^\nu A_{i'}^\mu A_{j'\nu} - \frac{g}{2} F_{jik} \bar{\Psi}_j \Gamma^\mu \Psi_k = 0, \quad (11)$$

$$F_{ijk} A_{j\mu} \Gamma^\mu \Psi_k = 0, \quad (12)$$

where

$$G_{ij;i'j'} = f_{ijk} f_{i'j'k'} g_{kk'} + \kappa^2 e_{ij} e_{i'j'}, \quad (13)$$

$$F_{jik} = (i f_{jkl} + d_{jkl})(g_{il} + i \kappa e_{il}) - (i f_{jil} + d_{jil})(g_{lk} + i \kappa e_{lk}), \quad (14)$$

We can write

$$F_{ijk} = i \tilde{f}_{ijk} + \tilde{d}_{ijk}, \quad (15)$$

Then the condition that the fermion term in the equation (11) be real is satisfied by d or f being symmetric or anti-symmetric respectively when we exchange the first and the last suffices and this leads to the condition stated under the equation (9). We define following variables:

$$A_{j,2c} + i A_{j,2c+1} \equiv \alpha_{j,c} = |\alpha_c| \beta_{j,c}, \quad (16)$$

with

$$\sum |\beta_{j,c}|^2 = 1, \quad (17)$$

and

$$A_{j,1} \pm A_{j,0} = \alpha_{j,0}^\pm = |\alpha_0| \beta_{j,0}^\pm, \quad (18)$$

with

$$\beta_{j,0}^+ \beta_{j,0}^- = \pm 1 \quad (+ \text{ for space-like and } - \text{ for time-like vectot } A_\mu). \quad (19)$$

Then the equation (12) can be reduced to the following five equations (we write down the IIB case):

$$F_{ijk}(\alpha_{j,a} x_{k,a} - \alpha_{j,0}^+ x_k) = 0, \quad (20)$$

$$F_{ijk}(\alpha_{j,a}^* y_{k,a} - \alpha_{j,0}^+ y_k) = 0, \quad (21)$$

$$F_{ijk} \left\{ \frac{1}{2} (\alpha_{j,a}^* y_{k,b} - \alpha_{j,b}^* x_{k,a}) - \alpha_{j,0}^+ x_{k;a,b} - \varepsilon_{abcd} \alpha_{j,c} y_{k,d} \right\} = 0, \quad (22)$$

$$F_{ijk}(\alpha_{j,a} y_k + \alpha_{j,0}^- y_{k,a} + \varepsilon_{abcd} \alpha_{j,b}^* x_{k;c,d}) = 0, \quad (23)$$

$$F_{ijk}(\alpha_{j,a}^* x_k - 2\alpha_{j,b} x_{k;a,b} + \alpha_{j,0}^- x_{k,a}) = 0, \quad (24)$$

Here the repeated suffices must be summed over. (11) can be written as (again IIB case)

$$4G_{ij;i'j'} A_j^\nu A_{i'}^\mu A_{j'\nu} - \frac{g}{2} F_{jik} \langle \Gamma^\mu \rangle_{j,k} = 0, \quad (25)$$

with

$$\langle \Gamma^0 \rangle_{j,k} = x_j^* x_k + 2x_{j;a,b}^* x_{k;a,b} + y_j^* y_k + x_{j,a}^* x_{k,a} + y_{j,a}^* y_{k,a}, \quad (26)$$

$$\langle \Gamma^1 \rangle_{j,k} = x_j^* x_k + 2x_{j;a,b}^* x_{k;a,b} - (y_j^* y_k + x_{j,a}^* x_{k,a} + y_{j,a}^* y_{k,a}), \quad (27)$$

$$\begin{aligned} \langle \Gamma^{2a} \rangle_{j,k} &= -x_j^* x_{k,a} - x_{j,a}^* x_k - y_j^* y_{k,a} - y_{j,a}^* y_k \\ &\quad - 2x_{j;a,b}^* x_{k,b} - 2x_{j,b}^* x_{k;a,b} - \varepsilon_{abcd} (x_{j;c,b}^* y_{k,d} + y_{j,d}^* x_{k;c,b}), \end{aligned} \quad (28)$$

$$\begin{aligned} \langle \Gamma^{2a+1} \rangle_{j,k} &= -i \{ x_j^* x_{k,a} - x_{j,a}^* x_k - y_j^* y_{k,a} + y_{j,a}^* y_k \\ &\quad - 2x_{j;a,b}^* x_{k,b} + 2x_{j,b}^* x_{k;a,b} - \varepsilon_{abcd} (-x_{j;c,b}^* y_{k,d} + y_{j,d}^* x_{k;c,b}) \}, \end{aligned} \quad (29)$$

In writing down these equations, the Grassmannian factor $\zeta^* \zeta$ or $\zeta^T \zeta$ (in case of IIA) is absorbed in the coupling constant g . In fact, the easiest way to get rid of Grassmannian factor is to assume that all A^μ contain a factor $(\zeta^* \zeta)^{2/3}$ (IIB) or $(\zeta^T \zeta)^{2/3}$ (IIA). Finally the equation (25) will become the following two sets of equations:

$$\rho_i^\pm |\alpha_0|^3 + \left(\sum_a |\alpha_a|^2 \sigma_{i,a}^\pm \right) |\alpha_0| = \frac{g}{4} \gamma_i^{0\pm} |\lambda|^2, \quad (30)$$

$$\sigma_{i;c,c} |\alpha_c|^3 + \left(\sum_{a \neq c} \sigma_{i;a,c} |\alpha_a|^2 + \rho_{i,c} |\alpha_0|^2 \right) |\alpha_c| = \frac{g}{4} \gamma_i^c |\lambda|^2, \quad (31)$$

where

$$\gamma_i^{0\pm} = \tilde{\gamma}_i^1 \pm \tilde{\gamma}_i^0, \quad (32)$$

$$\gamma_i^c = \tilde{\gamma}_i^{2c} + i \tilde{\gamma}_i^{2c+1}, \quad (33)$$

with

$$F_{jik}\langle\Gamma_{j,k}^\mu\rangle=\tilde{\gamma}_i^\mu|\lambda|^2, \quad (34)$$

and

$$\rho_i^\pm = G_{ij;i'j'}(\beta_{j,0}^+\beta_{j',0}^- + \beta_{j,0}^-\beta_{j',0}^+)\beta_{i',0}^\pm, \quad (35)$$

$$\sigma_{i,a}^\pm = G_{ij;i'j'}(\beta_{j,a}\beta_{j',a}^* + \beta_{j,a}^*\beta_{j',a})\beta_{i',0}^\pm, \quad (36)$$

$$\rho_{i,c} = G_{ij;i'j'}(\beta_{j,0}^+\beta_{j',0}^- + \beta_{j,0}^-\beta_{j',0}^+)\beta_{i',c}, \quad (37)$$

$$\sigma_{i,a,c} = G_{ij;i'j'}(\beta_{j,a}\beta_{j',a}^* + \beta_{j,a}^*\beta_{j',a})\beta_{i',c}, \quad (38)$$

The equations (30) and (31) have the desired form of elliptic curves. They must give the same value of $|\alpha_0|$ or $|\alpha_c|$ for the i -dependent coefficients. The consistency condition can be obtained in the following way: We first solve the equations for some value of i and substitute this back to equations (30) and (31). Then we have

$$|\alpha_0| = \wp(\omega) \quad \text{with} \quad \wp'(\omega) = 0, \quad (39)$$

and

$$|\alpha_c| = \wp_c(\omega_c) \quad \text{with} \quad \wp'_c(\omega_c) = 0, \quad (40)$$

where \wp and \wp_c are Weierstrass functions. The equations (30) and (31) become respectively,

$$4\wp^3(\omega)\rho_i^\pm + 4\sum_c \wp_c^2(\omega_c)\sigma_{i,c}^\pm\wp(\omega) = g\gamma_i^\pm, \quad (41)$$

$$4\sum_c \wp_c^3(\omega_c)\sigma_{i,c,a} + 4\wp^2(\omega)\wp_a(\omega_a)\rho_{i,a} = g\gamma_i^a, \quad (42)$$

These two equations are regarded as the consistency conditions but they themselves are a kind of coupled elliptic curves. My proposal is to define all these equations (coefficients and the solutions) on the field of rational numbers. The complex numbers must have real and imaginary parts which are rational respectively.

The 3-brane case can be further reduced to a simpler form as follows: This case is defined by

$$A^\mu \neq 0 \quad \text{only for} \quad \mu = 0, 1, 2, 3,$$

$$x_{j;a,b} = 0,$$

$$x_{j,a} \neq 0 \quad \text{only for} \quad a = 1,$$

$$y_{j,a} \neq 0 \quad \text{only for} \quad a = 1,$$

$$x_j \neq 0, \quad y_j \neq 0$$

The equations (20) to (24) become,

$$F_{ijk}(\alpha_{j,1}x_{k,1} - \alpha_{j,0}^+x_k) = 0, \quad (43)$$

$$F_{ijk}(\alpha_{j,1}^*y_{k,1} - \alpha_{j,0}^+y_k) = 0, \quad (44)$$

$$F_{ijk}(\alpha_{j,1}y_k + \alpha_{j,0}^-y_{k,1}) = 0, \quad (45)$$

$$F_{ijk}(\alpha_{j,1}^*x_k + \alpha_{j,0}^-x_{k,1}) = 0, \quad (46)$$

The symmetry of the equation under;

$$(x_j, x_{j,1}) \rightarrow (y_j^*, y_{j,1}^*),$$

makes it possible to set $(y_j^*, y_{j,1}^*) = (\theta x_j, \theta x_{j,1})$. Then we have,

$$\begin{aligned}\gamma_i^{0+} &= 2F_{jik}(1 + |\theta|^2)x_j x_k^*, \\ \gamma_i^{0-} &= 2F_{jik}(1 + |\theta|^2)x_{j,1} x_{k,1}^*, \\ \gamma_{i,1} &= 2F_{jik}(1 + |\theta|^2)x_{j,1}^* x_k,\end{aligned}$$

It can be shown in this case that,

$$\begin{aligned}|\alpha_0| &= 2\pi^2 \eta(\tau)^4, \\ |\alpha_1| &= 2\pi^2 \eta(\tau_1)^4,\end{aligned}$$

where $\eta(\tau)$ is the Dedekind modular form. Instead of (41) and (42) we have,

$$4\eta(\tau)^{12} \rho_i^\pm + 4\eta(\tau_1)^8 \eta(\tau)^4 \sigma_{i,1}^\pm = \frac{1}{8\pi^6} g \gamma_i^\pm, \quad (47)$$

$$4\eta(\tau_1)^{12} \sigma_{i,1,1} + 4\eta(\tau)^8 \eta(\tau_1)^4 \rho_{i,1} = \frac{1}{8\pi^6} g \gamma_{i,1}, \quad (48)$$

One might think that it may be possible to get a one-brane equation by putting

$$A^2 = A^3 = 0 \text{ and } x_{j,1} = y_{j,1} = 0 \text{ in the above equations.}$$

But, unfortunately, we can easily prove that the resulting set of equations do not have a solution. One-brane case must be defined in a different way. Another case is that of M2-brane. We write down the equations when we have the extended Pauli spin.

$$\vec{\alpha}^{0+} \times \vec{x} + \vec{A}^{11} \times \vec{x} = 0, \quad (49)$$

$$\vec{\alpha}^{0+} \times \vec{x} + \vec{A}^{11} \times \vec{x} = 0, \quad (50)$$

$$\vec{\alpha}^{0+} \times \vec{x}_{a,b} + \vec{A}^{11} \times \vec{x}_{a,b} = 0, \quad (51)$$

$$\vec{\alpha}^{0+} \times \vec{x}_{a,b} + \vec{A}^{11} \times \vec{x}_{a,b} = 0, \quad (52)$$

$$\begin{aligned}2\vec{\alpha}^{0+} \times (\vec{\alpha}^{0+} \times \vec{\alpha}^{0-}) + 4\vec{A}^{11} \times (\vec{\alpha}^{0+} \times \vec{A}^{11}) \\ + 2\kappa^2 \{ (\vec{\alpha}^{0+} \times \vec{\alpha}^{0-} \cdot \vec{I}) \vec{\alpha}^{0+} + \vec{I} + (\vec{\alpha}^{0+} \times \vec{A}^{11} \cdot \vec{I}) \vec{A}^{11} \times \vec{I} \} = 4ig(\vec{x}_{a,b}^* \times \vec{x}_{a,b}),\end{aligned} \quad (53)$$

$$\begin{aligned}2\vec{\alpha}^{0-} \times (\vec{\alpha}^{0-} \times \vec{\alpha}^{0+}) + 4\vec{A}^{11} \times (\vec{\alpha}^{0+} \times \vec{A}^{11}) \\ + 2\kappa^2 \{ (\vec{\alpha}^{0-} \times \vec{\alpha}^{0+} \cdot \vec{I}) \vec{\alpha}^{0-} \times \vec{I} + (\vec{\alpha}^{0-} \times \vec{A}^{11} \cdot \vec{I}) \vec{A}^{11} \times \vec{I} \} = -4ig(\vec{x}_{a,b}^* \times \vec{x}_{a,b}),\end{aligned} \quad (54)$$

$$\begin{aligned}2\vec{\alpha}^{0-} \times (\vec{A}^{11} \times \vec{\alpha}^{0+}) - 2\vec{\alpha}^{0+} \times (\vec{\alpha}^{0-} \times \vec{A}^{11}) \\ + 2\kappa^2 \{ (\vec{\alpha}^{0-} \times \vec{\alpha}^{0+} \cdot \vec{I}) \vec{\alpha}^{0-} \times \vec{I} + (\vec{\alpha}^{0-} \times \vec{A}^{11} \cdot \vec{I}) \vec{A}^{11} \times \vec{I} \} = ig(\vec{x}_{a,b}^* \times \vec{x}_{a,b} + \vec{x}_{a,b}^* \times \vec{x}_{a,b}).\end{aligned} \quad (55)$$

Equations (49)-(52) should be coupled with the equations (53)-(55).

4 Gravity

We might define the gravitational field (metric tensor) as a functional of the space-time coordinate. But by doing so we must introduce a new action to determine the form of the functional. This is in a way inappropriate since we want the original matrix action to be able to determine everything.

An elementary way to define the metric tensor is the following: First we define the operator ∇ which operates on any function defined on the space of clusters by,

$$\nabla f(n) = f(n) - f(n-1). \quad (56)$$

Then,

$$ds(n)^2 = g_{\mu,\nu}^v(n) Tr(\nabla A^\mu(n) \nabla A^\nu(n)) = Tr(g_{\mu,\nu}(n) \nabla A^{v,\mu}(n) \nabla A^{v,\nu}(n)). \quad (57)$$

Here the notation v stands for the vacuum. The vacuum solution $A^{v,\mu}(n)$ which corresponds to the solution $A^\mu(n)$ can be defined by taking the limit $g \rightarrow 0$ and $f_{ijk} \rightarrow 0$ in the solution $A^\mu(n)$. The latter limit can be taken by multiplying a small parameter to f_{ijk} and taking it to zero.

The general covariance is defined as the transformation property of g under the interchange of clusters: For $n' = n'(n)$, we have,

$$Tr(g'_{\mu,\nu}(n') \nabla A^{v,\mu}(n') \nabla A^{v,\nu}(n')) = Tr(g_{\mu,\nu}(n) \nabla A^{v,\mu}(n) \nabla A^{v,\nu}(n)). \quad (58)$$

The vacuum metric $g_{\mu,\nu}^v$ could be just the flat metric $\eta_{\mu,\nu}$, but it could also be a metric of the de Sitter or the anti-de Sitter space. Either way, we assume it is proportional to the unit matrix.

The connection is defined as

$$A_{\mu\nu,\rho\kappa} = \frac{1}{4} [\{g_{\mu\nu}(n) \nabla g_{\rho\kappa}(n) - g_{\rho\nu}(n) \nabla g_{\mu\kappa}(n)\} - \{\mu, \nu \leftrightarrow \rho, \kappa\}]. \quad (59)$$

The justification comes from the idea of non-commutative geometry [14]. In this particular case, we treat ∇ as the Dirac operator defined on the functional space of $g_{\mu\nu}$. We have the following identities,

$$A_{\mu\nu,\rho\kappa}(n) = A_{\rho\kappa,\mu\nu}(n), \quad A_{\mu\nu,\rho\kappa}(n) = -A_{\rho\nu,\mu\kappa}(n) = A_{\mu\kappa,\rho\nu}(n).$$

The curvature is defined again loosely following non-commutative geometry,

$$R_{\mu\nu,\rho\kappa}(n) \equiv \nabla A_{\mu\rho,\nu\kappa}(n) + \frac{1}{2} (A_{\mu\rho,\nu'\kappa'}(n) A^{\nu'\kappa'}_{\nu\kappa}(n) - A_{\nu\rho,\nu'\kappa'}(n) A^{\nu'\kappa'}_{\mu\kappa}(n)). \quad (60)$$

We have the following identities,

$$R_{\mu\nu,\rho\kappa}(n) = -R_{\nu\mu,\rho\kappa}(n) = -R_{\mu\nu,\kappa\rho}(n).$$

We do not have the cyclic identity for $R_{\mu\nu,\rho\kappa}(n)$. We can, therefore, define the Ricci tensor uniquely but it is not necessarily symmetric under the interchange of suffices. The vacuum curvature is assumed to satisfy (neglecting the symbol v),

$$R_{\nu\kappa}(n) = g^{\mu\rho}(n) R_{\mu\nu,\rho\kappa}(n) = \lambda g_{\nu\kappa}(n). \quad (61)$$

This equation can be easily solved by the following ansatz,

$$\nabla g_{\mu\nu}(n) = g_{\mu\nu}(n) - g_{\mu\nu}(n-1) = \delta g_{\mu\nu}(n),$$

where δ is a certain rational number. The $g_{\mu\nu}(n)$ can easily be solved,

$$g_{\mu\nu}(n) = \frac{1}{(1-\delta)^{n-1}} g_{\mu\nu}(1). \quad (62)$$

Equation (61) is reduced to,

$$\lambda = \frac{N-1}{8} [(N+6)\delta^2 - 4\delta^3]. \quad (63)$$

N is the dimension of the space-time.

These equations give a new interpretation to the cosmological constant λ as follows: When λ is small we have,

$$g_{\mu\nu}(n) = \exp\{(n-1)\delta\} g_{\mu\nu}(1). \quad (64)$$

Assuming that δ is a negative number, this equation shows that $1/|\delta|$ is the effective number of vacuum space-time points. The distance between the points with the cluster number larger than $1/|\delta|$ becomes small compared to the Planck scale and so the points become undistinguishable.

In the case of our universe, $\lambda \approx 10^{-120}$ which gives $1/|\delta| \approx 10^{60}$. The curvature size of the universe is 10^{28} cm and if we divide this by the Planck length 10^{-33} cm, we get 10^{61} . Since our space-time is 4-dimensional we have 10^{244} as the number of points of the current universe. The corresponding vacuum seems to have only one-dimensional degrees of freedom.

5 (First) Quantization and the Non-commutative Geometry

The matrix model is equipped with its algebra and the Hilbert space as the representation space of the algebra. The (first) quantization can be done utilizing this Hilbert space. The missing operator is the canonical conjugate variable P_μ to A^μ and Q_α to Ψ_α with the commutation relation,

$$[P_\mu, A^\nu] = -i\delta_\mu^\nu,$$

and

$$\{\Psi_\alpha, \overline{Q}_\beta\} = \{\overline{\Psi}_\alpha, Q_\beta\} = i\delta_{\alpha\beta}.$$

If we interpret the algebra and the Hilbert space to be that of non-commutative geometry, the missing operator to complete the Conne's triple spectra [13] is the Dirac operator. It is natural to assume that they are also given by P_μ and Q_β . In this interpretation, the solution of the equations we have been discussing is not a classical solution but a quantum one with A^μ and Ψ_α diagonalized up to the clusters.

We can define the gauge fields in the space of matrix algebra using the standard prescription of non-commutative geometry [13]. The relation of gravity defined in the last section and the one defined in this way is not clear at this time.

P_μ and Q_β have the other roles. The former is the operator for the translational invariance of the Lagrangian (5),

$$\begin{aligned} A^\mu &\longrightarrow A^\mu + a^\mu I, \\ \Psi &\longrightarrow \Psi. \end{aligned}$$

The latter is the operator for one of the supersymmetry transformations defined in [2],

$$\begin{aligned} \Psi &\longrightarrow \Psi + \xi I, \\ A^\mu &\longrightarrow A^\mu. \end{aligned}$$

This corresponds to the second supersymmetry defined by IKKT as $\delta^{(2)}$ [2].

The supersymmetry operator corresponding to $\delta^{(1)}$ of IKKT can be defined as follows,

$$\begin{aligned} Q_\alpha^{(1)} &= (\Gamma_\mu \Psi)_\alpha P^\mu - \frac{1}{2} [A_\mu, A_\nu] (\Gamma^{\mu\nu} Q)_\alpha, \\ \overline{Q}_\alpha^{(1)} &= -(\overline{\Psi} \Gamma_\mu)_\alpha P^\mu - \frac{1}{2} [A_\mu, A_\nu] (\overline{Q} \Gamma^{\mu\nu})_\alpha. \end{aligned}$$

We then get the following anti-commutation relations for these operators,

$$\begin{aligned} \{Q_\alpha^{(1)}, Q_\beta^{(1)}\} &= \{\overline{Q}_\alpha^{(1)}, \overline{Q}_\beta^{(1)}\} = 0, \\ \{\overline{Q}_\alpha^{(1)}, Q_\beta^{(1)}\} &= P^\rho [A_\mu, A_\nu] (\delta_\rho^\nu P^\mu - \delta_\rho^\mu P^\nu). \end{aligned}$$

Defining the gauge fields including the gravity using Dirac operators P^μ and Q_α will be discussed in a future publication.

6 Conclusions

The proposal is made that the space-time is constructed out of integers or rational numbers rather than being a continuum. This is based on taking the matrix model as the fundamental theory of space-time as it is. The proposal is substantiated by observing that the reduced matrix model equations can be thought of as the elliptic curves which play the central role in number theory. The whole machinery of the number theory developed over the centuries could be incorporated into the understanding of our space-time.

The space-time coordinates A_μ and Ψ_α are mutually non-commuting but we assume that they can be partially diagonalized up to certain clusters. We are then left with the equations of finite sized matrices and they form a system of coupled elliptic curves.

The metric tensor for this space-time can be calculated using the solution and the prescription described in section 4. The gravity is completely determined by the original reduced matrix model equations in this sense. It is necessary to define the vacuum metric for this purpose. It could be just a flat metric but it could also have a finite cosmological constant when the number of effective space-time points is finite. The latter is given by the inverse of the former thus providing a new meaning to the cosmological constant. The fact that our space-time seems to have a finite cosmological constant corresponds to the fact that it is composed of finite number of effective space-time points.

The (first) quantized system of the matrix model can be understood within the framework of non-commutative geometry. The fact that the matrix model is equipped with the algebra and the representation space allows us to introduce a canonical variable as the Dirac operator in the non-commutative geometry. This possibly provides us an alternative way to define the gravity but it starts with the level of a connection or a gauge field. Our naive definition of the metric tensor must be related to the latter definition in a certain way.

Non-commutative geometry itself is very much related to the number theory but it is outside of the scope of the present work.

Finally I will list some of the problems for the future investigations:

- (1) Can the non-commutative definition of gauge theory lead to the $E(8) \times E(8)$ theory?
- (2) The discussions of section 2 suggest that we may need the non-abelian Galois extension of the field. What is the physical meaning of the Galois group?
- (3) Is it possible to understand our assertion that equation (1) should be modular within the framework of Langlands philosophy?
- (4) Is it possible to incorporate all the geometrical concept related to the string theory into the number theory formulation of the matrix model?

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References

- [1] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, “*M theory as a matrix model: A conjecture*,” Phys. Rev. **D55** (1997) 5112; hep-th/9610043.
- [2] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, Nucl. Phys. **B498** (1997) 467.
- [3] T. Eguchi and H. Kawai, “*Reduction of Dynamical Degrees of Freedom in the Large N Gauge Theory*”, Phys. Rev. Lett. **B48** (1982) 1063
- [4] See, for example, Yu. I. Manin and A. A. Panchishkin, “*Introduction to Modern Number Theory*”, Second corrected printing, Springer 2000, Page 321.
- [5] N. Seiberg and E. Witten, “*Electric-magnetic duality, monopole condensation and confinement in $N = 2$ supersymmetric Yang-Mills theory*”, Nucl. Phys. **B426**, 19 (1994); [Erratum-ibid. **B430**, 485 (1994)]; arXiv:hep-th/9407087.

- [6] More than three dozen papers are listed when a key word “elliptic curve” is put into the search box in the section of high energy theory in arXiv e-print library.
- [7] G. Shimura and Y. Taniyama, “*Complex Multiplication of Abelian Varieties and Its Application to Number Theory*”, Tokyo: Mathematical Society of Japan, 1961; G. Shimura, “*Introduction to the Arithmetic Theory of Automorphic Functions*”, Princeton University Press, 1971. See also reference [4], page 320, 330, 334.
- [8] A. Wiles, “*Modular Elliptic-Curves and Fermat’s Last Theorem*”, Ann. Math. 141 (1995) 443-551; R. Taylor. and A. Wiles, “*Ring-Theoretic Properties of Certain Hecke Algebras*”, Ann. Math. 141 (1995) 553-572.
- [9] F. Conrad Diamond and R. Taylor, “*Modularity of Certain Potentially Barsotti-Tate Galois Representations*”, J. Amer. Math. Soc. 12 (1999) 521-567;
H. Darmon, “*A Proof of the Full Shimura-Taniyama-Weil Conjecture is Announced*”, Not. Amer. Math. Soc. 46 (1999) 1397-1406.
- [10] A. Kapustin and E. Witten, “*Electric-Magnetic Duality and the Geometric Langlands Program*”, Communications in Number Theory and Physics, Vol.1, No1. (2007) 1-236.
- [11] E. Frenkel, “*Lectures on the Langlands Program and Conformal Field Theory*”, arXiv.hep-th/0512172.
- [12] J. Polchinski, *String Theory*, Cambridge University Press, 1998, Volume II, Appendix B.
- [13] J. Bagger and N. Lambert, “*Modeling Multiple M2’s*”, Phys. Rev. **D75** (2007) 045020, arXiv.hep-th/0611108.
- [14] A. Conne, *Noncommutative Geometry*, Academic Press, 1994; M. Marcolli, Lectures on Arithmetic Noncommutative Geometry, ArXiv:math/0409520.